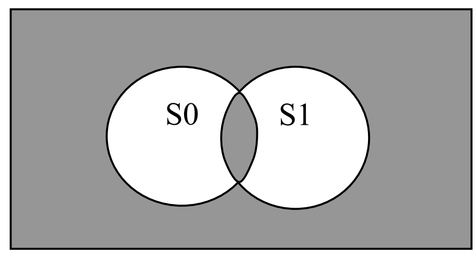
Chapter 1 Markov

Markov Analysis

In this section we will talk briefly about markov chain analysis of probability states.

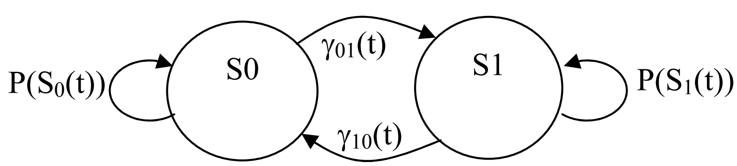
Figure 1 shows a Venn diagram of two probability sets plus all probabilities that are not contained in either set. Each set represents the probability that our system is in a given state. The shaded intersection is interpreted as no members of S0 are contained in S1 and visa-versa. Also, there are no states outside S0 and S1, they are complete.
However, it is possible to instantaneously flip between the states.



In a markov diagram, circles represent mutually exclusive states that the system can attain, denoted S0, S1, S2…Sn
Arrows represent the transitions out-of or in-to a state.

The probability of being in a state is equal to 1 minus the probability of the sum of all exits from the state.

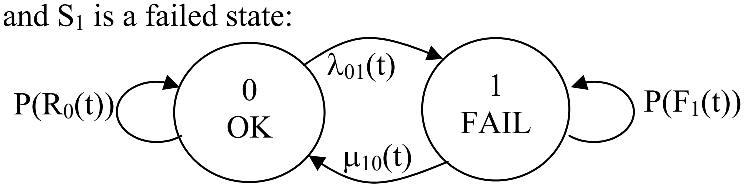
$$P(S_n(t)) = 1 - \sum_{n=0}^{m} \gamma_{nm}(t) \tag{1.1}$$



In the diagram above, the probability of being in state S0 at time *t* is:

$$P(S_0(t)) = 1 - \gamma_{01}(t)$$

In reliability analysis, states represent success and failure states of a system. Transitions from a lower to a higher state are failure probabilities $\lambda(t)\Delta t$, and transitions from higher to lower states are considered repair rates, $\mu(t)$.
With that in mind the figure can be redrawn as a reliability model where S_0 is the fully operational state and S_1 is a failed state:



The English interpretation of this diagram is:
Given the system starts in state 0;
The probability of transitioning to the failed state is the failure rate multiplied by the time interval $\lambda(t)\Delta t$. Therefore the probability of not transitioning is $1-\lambda(t)\Delta t$.
Once the system has transitioned to state 1, the probability of transitioning back to state S0 is the probability that it will be repaired within Δt, repair rate $\mu_{10}(t)\ \Delta t$. Therefore the probability of staying in state S1 is $1-\mu_{10}(t)\ \Delta t$.

Reminder: when $\lambda(t)\Delta t<<1$,
$1-e^{-\lambda(t)\Delta t}\approx\lambda t$

In order to solve markov chains it is necessary to use linear algebra to solve for the probability of being in any state at time t. Using this method it is possible to do two things:

1. Solve for the limiting, steady state probabilities for each state; and
2. Create a model that gives probability of failure as a function of time.

We start with the probabilities of being in a state or transition at any time t. This is the transition matrix. It describes the probability of transitioning from one state to another. The matrix is constructed by inserting the probability of the transition from row number to column number. The matrix diagonal () is the probability of being in a state. All others are transition from one state to another.

$$\begin{bmatrix} P_{1\to1} & P_{1\to2} & P_{1\to3} & P_{1\to n} \\ P_{2\to1} & P_{2\to2} & P_{2\to3} & P_{2\to n} \\ P_{3\to1} & P_{3\to2} & P_{3\to3} & P_{3\to n} \\ P_{n\to1} & P_{n\to2} & P_{n\to3} & P_{n\to n} \end{bmatrix}$$

Next we give starting conditions. This is the S matrix. If the system starts out fully repaired and perfectly operable, the starting matrix is:

$$S=\begin{bmatrix}1 & 0 & 0 & 0_n\end{bmatrix}$$

When we multiply the transition matrix by the starting matrix we get the matrix at t+dt.

Transition Matrix

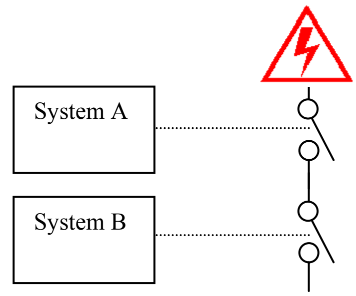
$$P = \begin{bmatrix} P(S_0) & \lambda_{01} & \dots & \lambda_{0n} \\ \mu_{10} & P(S_1) & & \\ \dots & & \dots & \\ \mu_{m0} & & & P(S_n) \end{bmatrix} \tag{1.2}$$

note that the sum of any row must add up to 1.

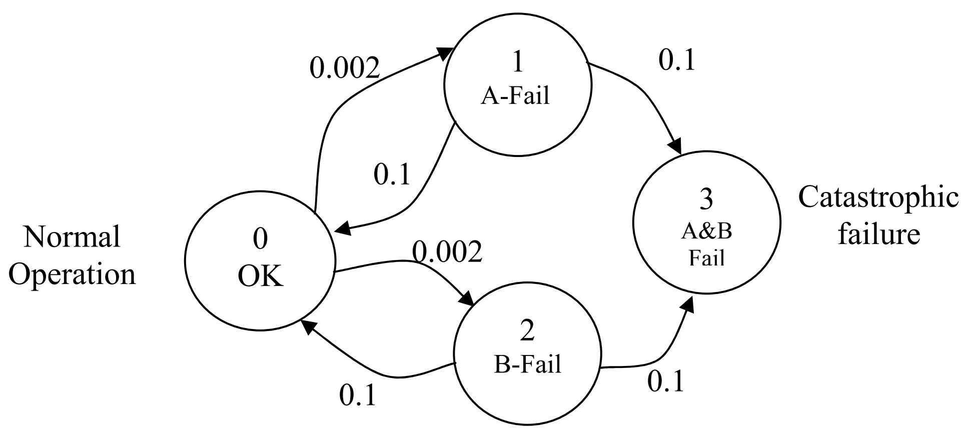
Example (Based on Goble Example 8-6, pp 171)

The independent and redundant safety system in figure has four operating states.

- All Systems operational
- System A failed but System B Operational
- System B failed but System A Operational
- And both systems failed.



Repair of the safety function is possible in the partially failed states. It is in the last state that the safety function is lost and an unmitigated accident is possible. At that point it is assumed that repair is not possible.
The markov model for this system is given below.



The transition matrix for the model is given below:

$$P = \begin{bmatrix} 0.996 & 0.002 & 0.002 & 0 \\ 0.1 & 0.899 & 0 & 0.001 \\ 0.1 & 0 & 0.899 & 0.001 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S^1 = S^0 \times P = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.996 & 0.002 & 0.002 & 0 \\ 0.1 & 0.899 & 0 & 0.001 \\ 0.1 & 0 & 0.899 & 0.001 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algebraic method of finding limited state probability

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jk} \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{j1} & \cdots & b_{jk} \end{pmatrix} \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{j1} & \cdots & c_{jk} \end{pmatrix}$$

$$aij = \sum_{k=1}^{n} b_{jk} c_{kj}$$

i is row and j is column

$$\begin{bmatrix} S_1^L & S_2^L \end{bmatrix} = \begin{bmatrix} S_1^{L-1} & S_2^{L-1} \end{bmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{bmatrix} S_1^L & S_2^L \end{bmatrix} = \begin{bmatrix} S_1^{L-1} & S_2^{L-1} \end{bmatrix}$$

$$S_1^L = aS_1^L + cS_2^L$$

$$S_1^L = aS_1^L + cS_2^L$$

$$S_1^L = \frac{cS_2^L}{(1-a)} .or. S_2^L = \frac{S_1^L(1-a)}{(1-c)}$$

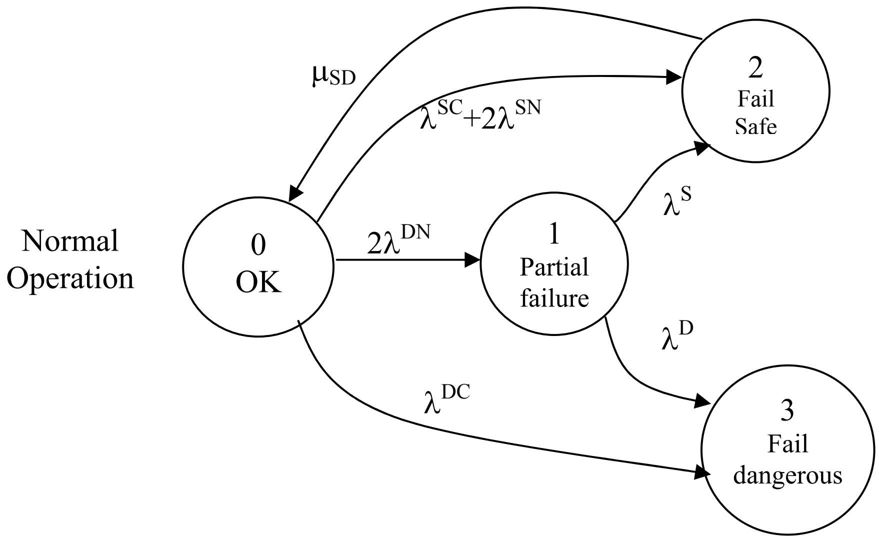
$$S_1^L = \frac{S_2^L(1-d)}{b} .or. S_1^L = \frac{bS_1^L}{(1-d)}$$

$$S_1 + S_2 = 1$$

$$S_1 + S_1 \frac{(1-a)}{c} = 1$$

$$S_1 = \frac{c}{c + (1-a)}$$

$$S_2 = 1 - S_1$$



$\lambda^{SN} = (1-\beta)\,\lambda^{S}$ **N**ormal (non-common cause) **s**afe failure
$\lambda^{SC} = \beta\lambda^{S}$ **C**ommon cause **s**afe failure rate
$\lambda^{DN} = (1-\beta)\,\lambda^{D}$ **N**ormal (non-common cause) **D**angerous failure
$\lambda^{DC} = \beta\lambda^{D}$ **C**ommon cause **D**angerous failure rate

[Goble pp 290]

$$P=\begin{bmatrix} 1-(\lambda^{SC}+2\lambda^{SN}+\lambda^{DC}+2\lambda^{DN}) & 2\lambda^{DN} & \lambda^{SC}+2\lambda^{SN} & \lambda^{DC} \\ 0 & 1-(\lambda^{S}+\lambda^{D}) & \lambda^{S} & \lambda^{D} \\ \mu_{SD} & 0 & 1-\mu_{SD} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P=\begin{bmatrix} .99981 & 0.00009 & 0.000095 & 0.000005 \\ 0 & 0.9999 & 0.00005 & 0.00005 \\ 0.041667 & 0 & 1-\mu_{SD} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$